Ergodic Theory - Week 5

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1 Birkhoff's pointwise ergodic theorem

P1. Let $u = [a_1, a_2, \ldots] \in (0,1)$, and let $\frac{p_n}{q_n}$ be its n-th convergent. Show that for any $n \in \mathbb{N}$

$$\Big|u - \frac{p_n}{q_n}\Big| > \frac{1}{q_n q_{n+2}}.$$

Solution: Since $\frac{p_n}{q_n} < u < \frac{p_{n+1}}{q_{n+1}}$ or $\frac{p_{n+1}}{q_{n+1}} < u < \frac{p_n}{q_n}$, it follows that

$$\Big|u - \frac{p_n}{q_n}\Big| = \Big|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\Big| - \Big|\frac{p_{n+1}}{q_{n+1}} - u\Big|.$$

Using the properties of the convergents, we have that

$$\Big|\frac{p_{n+1}}{q_{n+1}} - u\Big| < \frac{1}{q_{n+1}q_{n+2}},$$

and

$$\Big|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\Big| = \Big|\frac{p_{n+1}q_n - p_n q_{n+1}}{q_{n+1}q_n}\Big| = \frac{1}{q_{n+1}q_n}.$$

Therefore

$$\Big|u - \frac{p_n}{q_n}\Big| > \frac{1}{q_{n+1}q_n} - \frac{1}{q_{n+1}q_{n+2}} = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} = \frac{a_{n+2}}{q_n q_{n+2}} \geq \frac{1}{q_n q_{n+2}},$$

where we used that

$$q_{n+2} = a_{n+2}q_{n+1} + q_n.$$

P2. A real number $u = [a_1, a_2, \ldots] \in (0,1)$ is called *badly approximable* if there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$ Show that $u \in (0,1)$ is badly approximable if and only if there exists some $\epsilon > 0$ such that

$$\Big|u - \frac{p}{q}\Big| \geq \frac{\epsilon}{q^2}$$

for all rational numbers $\frac{p}{q}$.

Solution: ($\Longrightarrow$) Assume that $u \in (0,1)$ is badly approximable. Write $u = [a_1, a_2, \ldots]$, and let $\frac{p_n}{q_n}$ be its n-th convergent. Recall that $q_{n+1} = a_{n+1}q_n + q_{n-1}$, and thus by hypothesis

$$q_{n+1} \leq (M+1)q_n,$$

using the fact that $q_n \geq q_{n-1}$. Now, fix $n \in \mathbb{N}$ such that $q \in (q_{n-1}, q_n]$. We have that

$$\Big|\frac{p}{q} - u\Big| > \Big|\frac{p_n}{q_n} - u\Big|,$$

since p_n/q_m is the best rational approximation. Thus, by problem 2:

$$\left|\frac{p}{q} - u\right| > \left|\frac{p_n}{q_n} - u\right| > \frac{1}{q_n q_{n+2}} \geq \frac{1}{(M+1)^2 q_n^2} \geq \frac{1}{(M+1)^4 q_{n-1}^2} \geq \frac{1}{(M+1)^4 q^2}.$$

($\Longleftarrow$) Conversely, assume that for some $\epsilon > 0$,

$$\left|u - \frac{p}{q}\right| \geq \frac{\epsilon}{q^2},$$

for all rational $\frac{p}{q}$. In particular, for all $n \geq 1$,

$$\frac{\epsilon}{q_n^2} \leq \left|u - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}},$$

where the last equality follows from the properties of convergents. We infer that

$$\epsilon^{-1} > \frac{q_{n+1}}{q_n} = \frac{a_{n+1} q_n + q_{n-1}}{q_n} = a_{n+1}(1 + \frac{q_{n-1}}{q_n}) \geq a_{n+1}.$$

for all $n \geq 2$. Taking $M \geq \text{máx}\{\epsilon^{-1}, a_1, a_2\}$, we reach our conclusion.

P3. (Strong Law of Large Numbers) Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and consider $(X_n)_{n\in\mathbb{N}}$ a sequence of independent and identically distributed (i.i.d.) Lebesgue integrable random variables, with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mu(\cdot) = \mathbb{P}(X_0 \in \cdot)$ the the probability distribution of X_i. Consider the space $X = \mathbb{R}^{\mathbb{N}}$ equipped with the product sigma-algebra $\mathcal{B}$, the product measure $\nu = \bigotimes_{n\in\mathbb{N}} \mu$, and the left shift $(x_i)_{i\in\mathbb{N}} \in X \to \sigma((x_i)_{i\in\mathbb{N}}) = (x_{i+1})_{i\in\mathbb{N}} \in X$.

(a) Justify that $(X, \mathcal{B}, \nu, \sigma)$ is a measure-preserving system.

Solution: The fact that $(X, \mathcal{B}(X), \nu)$ is a probability space comes from the definition of the product measure. Also, the shift is trivially measurable given that the preimage of cylinders are cylinders themselves. Thus, we only need to prove that σ preserves ν. Indeed, consider $C = \{x \in X : x_{i_1} \in A_1, \ldots, x_{i_N} \in A_N\}$ for $A_1, \ldots A_N \in \mathcal{B}(\mathbb{R})$. Then

$$\nu(\sigma^{-1}C) = \nu(\{x \in X : x_{i_1+1} \in A_1, \ldots, x_{i_n+1} \in A_N\}) = \prod_{j=1}^{N} \mu(A_i) = \nu(C).$$

Since σ preserves cylinder sets, it preserves all sets of $\mathcal{B}$ (since cylinder sets generate $\mathcal{B}$).

(b) Show that $(X, \mathcal{B}, \nu, \sigma)$ is mixing (see Exercise sheet 2 for the definition). Conclude, in particular, that it is ergodic.

Solution: Consider two cylinders $C_A = \{x \in X : x_1 \in A_1, \ldots, x_N \in A_N\}$ and $C_B = \{x \in X : x_1 \in B_1, \ldots, x_M \in B_M\}$. Consider $n > M$, and notice that

$$\begin{aligned}\nu(\sigma^{-n}C_A \cap C_B) &= \nu(\{x \in X : x_1 \in B_1, \ldots, x_M \in B_M, x_{n+1} \in A_1, \ldots, x_{N+n} \in A_N\}) \\ &= \Big(\prod_{i=1}^{M} \mu(B_i)\Big)\Big(\prod_{i=1}^{N} \mu(A_i)\Big) = \nu(C_A) \cdot \nu(C_B).\end{aligned}$$

Since we have $\nu(T^{-n}C_A \cap C_B) \to \nu(C_A) \cdot \nu(C_B)$ for cylinder sets, we infer that $\nu(\sigma^{-n}A \cap B) \to \nu(A)\nu(B)$ for all sets in the σ-algebra $\mathcal{B}$. Therefore, the system is mixing, which also implies that it is ergodic (a system is ergodic if and only if for any sets A, B we have

$$\frac{1}{N}\sum \mu(T^{-n}A \cap B) \to \mu(A)\mu(B)$$

and this is obviously true when the trasformation T is mixing).

(c) Conclude that

$$\frac{X_0 + \cdots + X_{N-1}}{N} \xrightarrow[N\to\infty]{} \mathbb{E}(X_0),\ \mathbb{P} - \text{a.e.}.$$

Solution: Notice that for $x = (x_n)_{n\in\mathbb{N}} \in X$

$$\frac{1}{N}\sum_{n=0}^{N-1} x_n = \frac{1}{N}\sum_{n=0}^{N-1} f(\sigma^n x),$$

where $f((x_n)_n) = x_0$ is the projection on the first coordinate. By Birkhoff's ergodic theorem, we get that for ν-almost every $x \in X$

$$\lim_N \frac{1}{N}\sum_{n=0}^{N-1} x_n = \int_X f d\nu = \int_{\mathbb{R}} x d\mu(x) = \mathbb{E}(X_0), \tag{1}$$

and for the definition of μ and ν, we get that

$$\frac{X_0 + \cdots + X_{N-1}}{N} \xrightarrow[N\to\infty]{} \mathbb{E}(X_0),\ \mathbb{P} - \text{a.e.}.$$

To see this, we can define the map $S : (\Omega, \mathcal{F}, \mathbb{P}) \to (X, \mathcal{B}(X), \nu)$ given by $w \to S(w) := (X_n(w))_{n\in\mathbb{N}}$. Notice that if $C = \{x \in X \colon x_{i_1} \in A_1, \dots, x_{i_N} \in A_N\}$ for $A_1, \dots A_N \in \mathcal{B}(\mathbb{R})$ then we have that

$$\mathbb{P}(S(w) \in C) = \mathbb{P}(X_{i_1} \in A_1, \dots, X_{i_N} \in A_N) = \prod_{j=1}^{N} \mathbb{P}(X_{i_j} \in A_j) = \nu(C).$$

Thus $\mathbb{P}(S^{-1}C) = \nu(C)$. We infer that $S^*\mathbb{P} = \nu$, given that we have the equality in an algebra that generates the σ-algebra.
Let Y be the set for which the convergence in (1) holds, so that $\nu(Y) = 1$. Then,

$$\mathbb{P}\left(\left\{w \in \Omega \colon \frac{X_0(w) + \cdots + X_{N-1}(w)}{N} \xrightarrow[N\to\infty]{} \mathbb{E}(X_0)\right\}\right) =$$
$$\mathbb{P}\left(\left\{w \in \Omega \colon \frac{f(S(w)) + f(\sigma(Sw)) + \cdots + f(\sigma^{n-1}(S(w)))}{N} \to \mathbb{E}(X_0)\right\}\right) =$$
$$\mathbb{P}\left(\{w \in \Omega \colon S(w) \in Y\}\right) = \nu(Y) = 1.$$